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What is a spacetime interval?

by @kjrunia

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Einstein and collaborators taught us that space and time are not fixed quantities. They can stretch and contract. They vary. There is one thing, though, that does not vary. It is the invariance of the spacetime interval.

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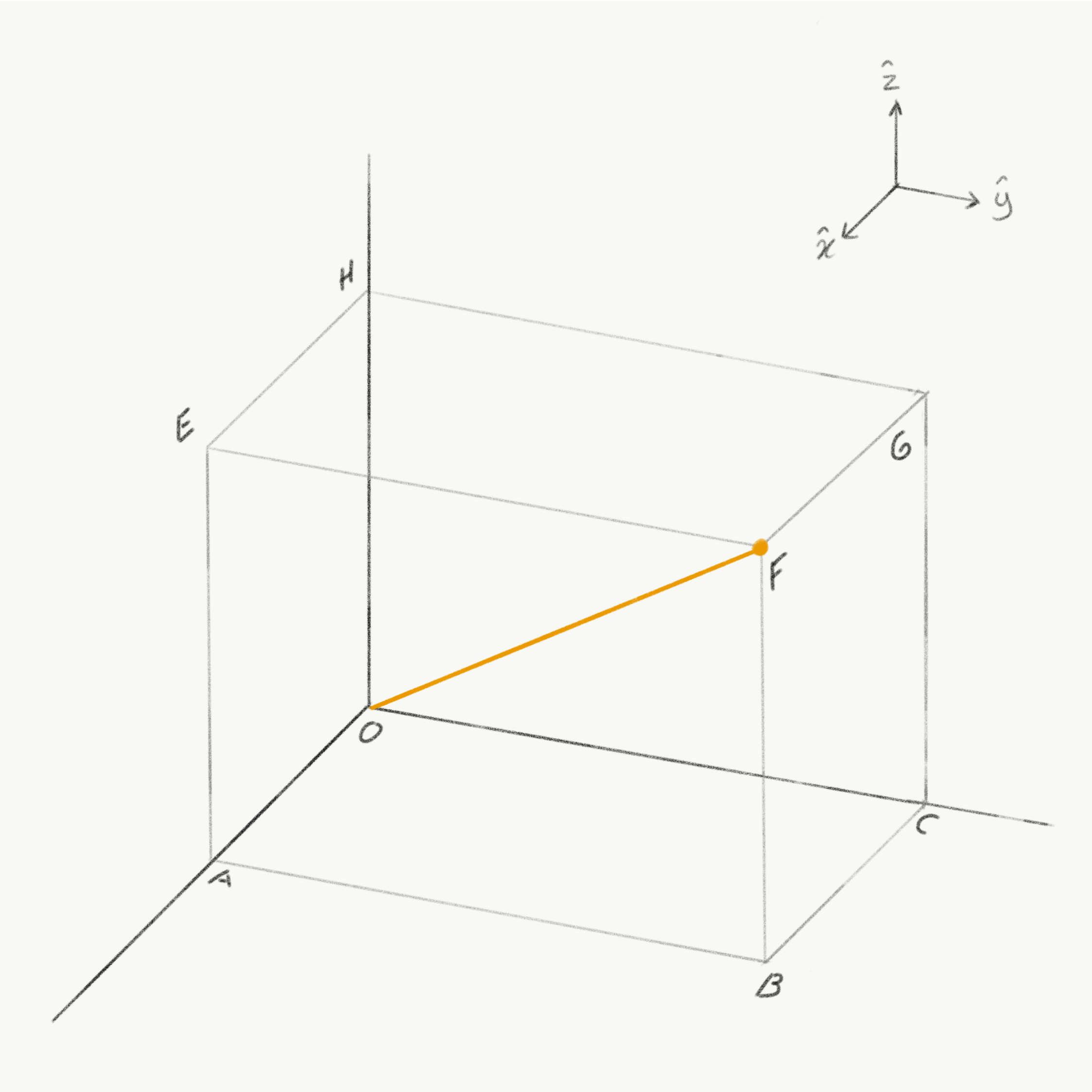

Figure 1: A photon travels from O to F in a three-dimensional space

1 Spatial interval

Suppose, a photon is emitted from origin O and travels to point F as depicted in Figure 1. Let us write down the expression for its distance-squared, $d(O,F)^2$, in terms of the other distances using the good old Pythagorean theorem:

$$d(O,F)^2 = d(O,A)^2 + d(A,B)^2 + d(B,F)^2.$$

We can also write the previous expression in terms of their distance from O. We then write the following:

$$\begin{aligned} F = (x,y,z), \quad O = (0,0,0), \\ d(O,F)^2 = (x-0)^2 + (y-0)^2 + (z-0)^2, \\ \therefore d(O,F)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2. \end{aligned} \tag{1}$$

As the axes of the space in Figure 1 are spatial and Euclidean, $d(O, F)^2$ is called a spatial or Euclidean interval. It can also be thought of as a rectangular cuboid represented by its space diagonal $d(O, F)$, tracing out a region of 3D space.

In the real world, to make sure we meet at the correct place, we could, for instance, give the following coordinates: 1 Einstein Drive, 2nd floor. Think of Einstein Drive as some place along the the x-axis (next to x-axis places like Battle Road, Mercer Road), number 1 as some place along the y-axis, and 2nd floor as some place along the z-axis.

What we still need, though, is an extra bit of information: when do we meet?

2 Time interval

Suppose, Figure 2 shows a timed series of our photon on its way to point F and beyond. It demonstrates that we live in a world where we do not just need three spatial coordinates, but also a time coordinate. It is only logical to not just tell the people you are supposed to meet, where in space you will be, but also when in time you will be there.

Our photon P flies through F at $t = 3$. This entails that the time coordinate of the event that the photon reaches F is

$$t_F = 3.$$

Assuming that at $t = 0$, photon P is at the origin,

$$t_O = 0,$$

then we can write for the temporal interval between the photon leaving O and reaching F:

$$\Delta t_{OF} = t_F - t_O = 3 - 0 = 3.$$

3 Time to distance unit conversion

As the previous two sections showed, we need four coordinates to describe an *event*, for instance, the event where photon P reaches F. The three spatial distances are measured in a unit of distance, usually, metres. The one temporal distance is not a distance in the traditional sense and is usually expressed in seconds. This makes it difficult to make sensible comparisons.

To convert the time-units to distance-units, we multiply by a constant of nature, the speed of light c, which, by Einstein's second postulate (Einstein, 1905), happens to be invariant: no matter which frame of reference you choose, the speed of light is constant. For a longer description of this conversion, read section 4.3 of Deriving the Lorentz transformations from a rotation of frames of reference about their origin with real time Wick-rotated to imaginary time. We conclude that our time interval becomes a temporal distance:

$$\Delta t \mapsto c\Delta t.$$

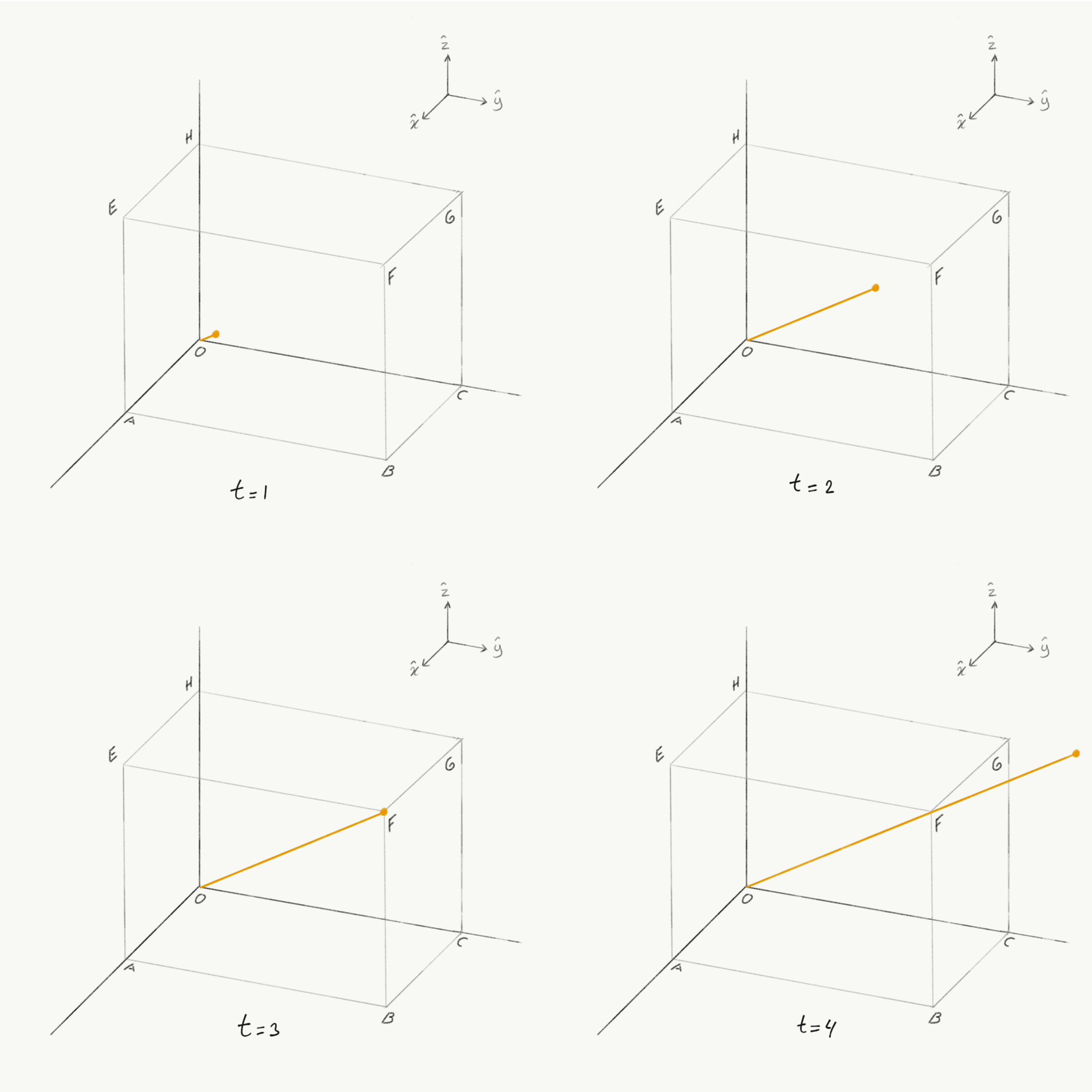

Figure 2: A photon travels from O to F in a three-dimensional space over a period of time

4 Spacetime interval

In Figure 3, we left out the *spatial* z-axis and replaced it with the *temporal* ct-axis (which is thus time expressed in distance-units) in order to make a comprehensible drawing on a flat surface. In reality, of course, the photon still moves in the z-direction as well. (We have thus not yet been successful to draw a four-dimensional object on a flat surface.) Mind the unit vector diagram top right and the points of distances Δx, Δy, and $c\Delta t$. Then think, really hard, of an added fourth spatial distance Δz, somewhere.

To calculate the spatial distance $d(O, F)$ for our photon, we look again at Equation (1):

$$d(O, F)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2. \quad (1)$$

Since we know that speed, in general, is calculated through $v = \Delta x/\Delta t$, where x is the travelled distance in one direction, and $v = c$ for our photon, we can write for the travelled distance of our photon from O to F:

$$\begin{aligned} d(O, F) &= v\Delta t, \\ d(O, F)^2 &= (v\Delta t)^2, \\ \therefore d(O, F)^2 &= (c\Delta t)^2. \end{aligned}$$

This is becoming interesting, since $(c\Delta t)^2$ is also (the square of) the *temporal* distance. If we substitute Equation (1) into this last equation, we get

$$(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = (c\Delta t)^2.$$

If we rearrange this a little bit, we get

$$-(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = 0.$$

While this may seem nice and simple, the question we should be asking is, what *is* zero? If we know the answer to that, we know the answer to what all the terms are on the left-hand side of the equals sign.

In physics and mathematics, whenever something equals zero, something special is going on: it may entail a certain system in a certain configuration that is stable, static even, it may point to constant motion, an energy well, an attractor, a root, a conservation law, homogeneity, or a minimum or maximum of some kind.

In general, it means that there is a certain kind of symmetry at play, which in turn means that something is conserved. There are beautiful, deep insights to be made as Emmy Noether showed us (Noether, 1918), and her genius deserves nothing less than an entire series of articles on their own.

However, for now, let us conclude that independent of which coordinate system one uses, rendering different values for Δx, Δy, Δz, and even Δt, as we have come to learn from the Lorentz transformations, *the sum of all these variances remains invariant.* The zero points to the fact that irrespective of its four moving parts—no matter what frame of reference you prefer—the resultant is a constant, i.e. invariant.

The quantity on the left-hand side has a name; it is called the spacetime interval and is denoted by $(\Delta s)^2$. The s stands for 'separation'. It is about the separation

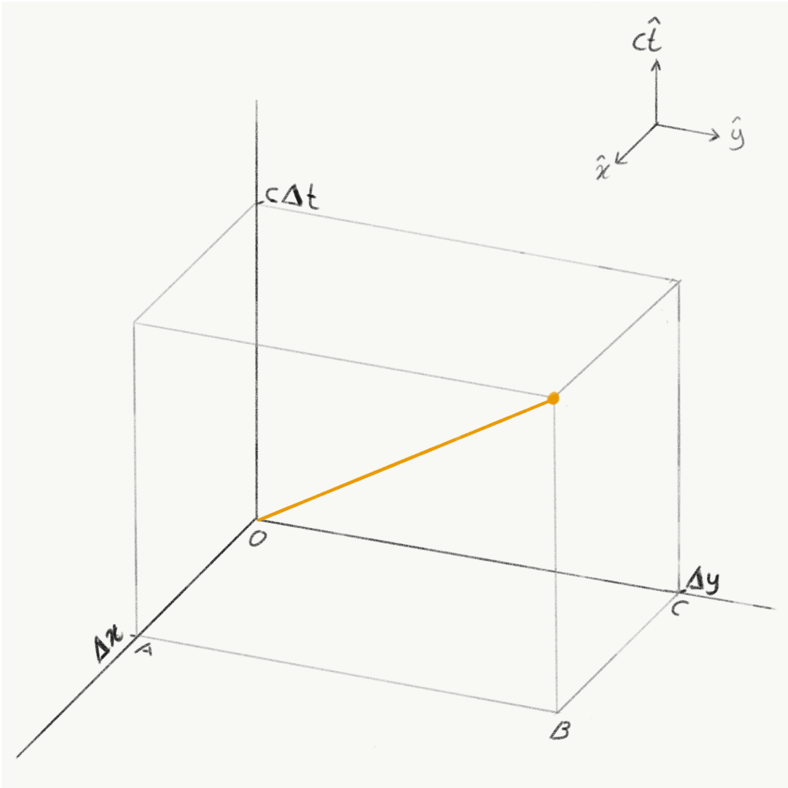

Figure 3: A spacetime diagram with two spatial dimensions and one temporal dimension.

between events. If we had used the word distance, it might have had inadvertently referred too much to a spatial distance, hence, we use separation, $(\Delta s)^2$. And so, the spacetime interval is usually written:

$$(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2.$$

The signs before the terms may have been flipped in some texts, but important to note is that, while time has been made comparable to space unit-wise by multiplication by c, you can still see that time has a special place in the interval of the fabric of the cosmos.

References

Einstein, A. (1905). Zur Elektrodynamik bewegter Körper, *Annalen der Physik* **322**(10): 891–921.

Noether, E. (1918). Invariante Variationsprobleme, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* **1918**: 235–257.
URL: *http://eudml.org/doc/59024*