
OPENCURVE

A BIT OF MATHS AND PHYSICS. FOR EVERYONE.

SIMPLE PROBLEMS ON RELATIVISTIC ENERGY AND MOMENTUM

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We will focus on a few simple problems where we will manipulate the equations for relativistic energy en momentum.

Contents

| | | |
|---|------------------------|---|
| 1 | Introduction | 2 |
| 2 | Problem I | 3 |
| 3 | Problem II | 3 |
| 4 | Problem III | 4 |



Proton with 300 GeV energy producing 26 charged particles in the 30 inch hydrogen bubble chamber at Fermilab. Source: Wikimedia Commons

1 Introduction

Einstein had shown that the Lorentz transformations were the correct way to switch between the coordinate systems of different frames of reference Einstein (1905). He also taught us that Newton's laws weren't at all proper relativistic laws. For instance, Newtonian momentum $\mathbf{p} = m\mathbf{v}$, and energy $E = mv^2/2$ were not at all accurate at high speeds.

Instead, we have all come to learn that the relativistic momentum is written as

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}}. \quad (1)$$

And that the correct, relativistic expression for total energy is

$$E_{\text{tot}} = \frac{mc^2}{\sqrt{1 - v^2/c^2}}. \quad (2)$$

We will solve the following problem set.

- I Proof, for a particle travelling at c , that the magnitude of the relativistic energy is given by $E = pc$.
- II Show that the energy-momentum relation for a particle with mass m travelling at speed v is correct and do mind it is **not** the famous $E = mc^2$ to which we are referring. The correct one, please.

- III Given that the mass of a proton is m_p , calculate its exact speed when its relativistic translational kinetic energy (which is the relativistic total energy minus its relativistic mass energy) is four times its relativistic mass energy.

2 Problem I

Since E is expressed in terms of p , we need to rewrite Eq. (1) by solving for m :

$$m = \frac{p\sqrt{1 - v^2/c^2}}{v}.$$

Note, we do not use the vector quantities, just the magnitudes. We can now proceed to substitute this into Eq. (2):

$$E_{\text{tot}} = \frac{\left(\frac{p\sqrt{1 - v^2/c^2}}{v}\right) c^2}{\sqrt{1 - v^2/c^2}}.$$

This reduces to

$$\begin{aligned} E_{\text{tot}} &= \frac{pc^2\sqrt{1 - v^2/c^2}}{v\sqrt{1 - v^2/c^2}}, \\ \therefore E_{\text{tot}} &= \frac{pc^2}{v}. \end{aligned} \quad (3)$$

As we are dealing with a particle travelling with speed c , we know $v = c$, rendering Eq. (3) to

$$\begin{aligned} E_{\text{tot}} &= \frac{pc^2}{c}, \\ \therefore E_{\text{tot}} &= pc. \end{aligned}$$

3 Problem II

The energy-momentum relation is

$$E_{\text{tot}}^2 = p^2c^2 + m^2c^4.$$

Substituting Eqs. (1) and (2), yields

$$\begin{aligned} \left(\frac{mc^2}{\sqrt{1 - v^2/c^2}}\right)^2 &= \left(\frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}}\right)^2 c^2 + m^2c^4, \\ \frac{m^2c^4}{1 - v^2/c^2} &= \frac{m^2v^2c^2}{1 - v^2/c^2} + m^2c^4, \\ (1 - v^2/c^2) \left(\frac{m^2c^4}{1 - v^2/c^2}\right) &= (1 - v^2/c^2) \left(\frac{m^2v^2c^2}{1 - v^2/c^2}\right) + (1 - v^2/c^2)m^2c^4, \\ m^2c^4 &= m^2v^2c^2 + m^2c^4 - m^2v^2c^4/c^2, \\ m^2c^4 - m^2c^4 &= m^2v^2c^2 - m^2v^2c^2, \\ 0 &= 0. \end{aligned}$$

Hence, for every value of m , p , and thus v , the relation is a true statement.

4 Problem III

The relativistic (total) energy is

$$E_{\text{tot}} = E_{\text{trans}} + E_{\text{mass}}.$$

If the relativistic translational kinetic energy is four times the relativistic mass energy, then we can write

$$E_{\text{trans}} = 4E_{\text{mass}}.$$

In our case, this then yields for the relativistic (total) energy:

$$E_{\text{tot}} = 4E_{\text{mass}} + E_{\text{mass}} = 5E_{\text{mass}}.$$

To calculate the proton's speed, we then write

$$\begin{aligned} \frac{m_p c^2}{\sqrt{1 - v^2/c^2}} &= 5E_{\text{mass}} = 5m_p c^2, \\ \frac{1}{\sqrt{1 - v^2/c^2}} &= 5, \\ \sqrt{1 - v^2/c^2} &= \frac{1}{5}, \\ 1 - \frac{v^2}{c^2} &= \frac{1}{25}, \\ \frac{v^2}{c^2} &= \frac{24}{25}, \\ v^2 &= \frac{24c^2}{25}, \\ \therefore v &= \sqrt{\frac{24c^2}{25}} = \frac{2\sqrt{6}c}{5}, \end{aligned}$$

which means that it zips at about 98% of the speed of light through the fabric of the cosmos.

References

Einstein, A. (1905). Zur Elektrodynamik bewegter Körper, *Annalen der Physik* **322**(10): 891–921.