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The riddle of birthdays

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Probabilities can be hard to grasp. For instance, what are the chances that among a birthday party's attendants two or more people will have their birthdays on the same day? Probably better than you might expect.

Since the day she was born, every year, my mother's birthday has been on 1 January. This year, she celebrated her twelfth jubilee year in a cosy party room filled with about fifty people.

Being the life and soul of any party, during my little talk, I presented the guests the fact that the probability of my mother's day of birth being 1 January equalled $1/365$. As most years consist of 365 days, I left leap years out of consideration. I also assumed a uniform distribution of birthdays in a year as this makes it easier to perform further calculations.

Then I asked what the probability was for my father to be born on 8 July, given that there 365 days to choose from. The answer was, again, 1 out of 365, or $1/365$. Of course, in this respect, a particular day is not more special than another particular day other than the cultural significance we assign to some.

Then I asked the crucial question: what is the probability that two or more people in this room share the same birthday? Of course, irrespective of their year of birth. It was purely about the day of the year.

In other words, there are 365 days in a year and we have 50 people whose birthdays have spread over those 365 days. What is the probability that two (or more) birthdays fall on the same day?

Here, I am increasing the party fun by putting forward a maths riddle during my little talk.

Sometimes, people think of an example with dice. Suppose, you have two dice. The probability of throwing a six is $1/6$, which is the same for throwing a six with the other dice. The chances of throwing sixes with both dice is, thus, $1/6 \times 1/6 = 1/36$. Logically, the probability is smaller than throwing a six with one dice. (Read When and why do you multiply probabilities?) Many people argue that the probability of two people having their birthday on 8 July, for instance, is therefore equal to $1/365 \times 1/365 = 1/133225$; which is, therefore, a very small probability. This would be in accordance with many people's intuition: it would be highly unlikely if two people, within a group of fifty people, would share their birthday, wouldn't it?

Others think of 50 marbles in a jar with 365 marbles. You draw one marble out of the jar en put it back again. You then shake the jar. Again, you draw a marble out of the jar. What is the probability you draw the same marble out of the jar? This way, people get the answer of 50/365.

But no, both strategies are incorrect. In reality, the probability is 96%, rounded to the nearest integer percentage. Therefore, I would want to bet a good bottle of wine on this.

The calculation

Often, in mathematics, it is easier to explore the opposite situation. Let us proceed accordingly. The reverse situation is that *no one* shares their birthday. Let us look at this more closely. What is the probability no one shares their birthday?

We have 365 days. We have 50 humans. What is the probability that human number 1 in the group has their birthday on *a* day? Mind the phrasing of the question. *A* day, not a *specific* day. Hence, we are not asking what the probability is of being born on, for instance, 8 July. We are asking ourselves what the probability is of human number 1 being born on *one of* those 365 days. Well, that is 365 out of 365 days, or $^{365}/_{365}$, or $365 : 365 = 1$, pr 100%.

Now, what is the probability that human number 2 in the group has their birthday on *a* day, but *not the same day* as human number 1 has theirs? Therefore, the possibilities for human number 2 to have their birthday are one fewer than 365, which is, perhaps not surprisingly, 364. Otherwise, both human number 1 and 2 could have had their birthdays on the same day. So, the probability that human number 2 is having their birthday on another day is 364 out of 365, or $^{364}/_{365}$, or $364 : 365 = 0.997\ldots$, or 99.7%.

And so, what is the probability that *both* have their birthday on different days? This is

$$\frac{365}{365} \times \frac{364}{365} = 1 \times 0.997\cdots = 0.997\ldots,$$

or 100% rounded to the nearest integer percentage. In other words, de chances are practically non-existent for them having their birthday on the same day.

Yet, what happens if we involve a third human? What is the probability for human number 3 having their birthday on a different day than those of humans number 1 and 2? This is then 363 out of 365, or $^{363}/_{365}$, or $363 : 365 = 0,994\ldots$, or 99.4%. And so, what are the odds that humans number 1, 2 and 3 all have their birthdays on different days? This is then

$$\frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} = \frac{132132}{133225} = 0.991\ldots,$$

or 99%, rounded.

Just to make sure, let us involve a fourth human. The probability for human number 4 to have their birthday on a different day than those of humans number 1, 2 and 3, is $^{362}/_{365}$, or $362 : 365 = 0.991\ldots$. And so, what is the probability that humans numbers 1, 2, 3 and 4 have their birthdays on different days? This is then

$$\frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} = \frac{47831784}{48627125} = 0.983\ldots,$$

or 98%, rounded. We can now clearly observe the decreasing probability of humans having their birthdays on different days with each addition of humans.

Imagine we would continue this process until human number 50. The probability that human number 50 has their birthday on any other day than the rest of the 49 preceding humans, then becomes $^{316}/_{365}$. So, the probability that *all* fifty humans have their birthdays on different days is calculable as follows:

$$\frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \cdots \times \frac{317}{365} \times \frac{316}{365} = 0.029\ldots,$$

that is, only 2.9%!

So, now we have our answer. The *opposite situation*, i.e. the probability that *not all* fifty humans have their birthdays on different days, is the reverse of 2.9% and this is 97.1%. Or, 97% rounded.

Among the fifty guests, no fewer than six people turned out to share their birthday. In other words, we found three 'pairs' sharing their birthday.

By the way, among a group of 23 people, the probability is already 0.504 (so, just over 50%) for two (or more) people to share their birthday. The odds grow favourably quickly.

Might you want to read more, this phenomenon rests on the pigeonhole principle or Dirichlet's box principle—this nineteenth century German mathematician was probably the first one to formalise it. Happy Googling! (Though we highly recommend DuckDuckGo.com.)